

Power Graph and Exchange Property for Resolving Sets

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Abstract

A formula for computing the metric dimension of a simple graph, having no singleton twin, is given. A sufficient condition for a simple graph to have the exchange property, for resolving sets, is found. Some families of power graphs of finite groups, having this exchange property, are identified. The metric dimension of the power graph of a dihedral group is also computed.

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1 Introduction

Let G be a finite group. An *undirected power graph* \mathcal{P}_G , associated to G , is a graph whose vertices are the elements of G , and there is an edge between two vertices x and y if either $x^m = y$ or $y^m = x$, for some positive integer m . The power digraph of G is the digraph $\overrightarrow{\mathcal{P}}_G$ with the vertex set G , and there is an arc from vertex x to y if $x^m = y$, for some positive integer m . The power digraphs were considered in [12, 13, 14]. Motivated by this, Chakrabarty, Ghosh, and Sen (see [6]) studied undirected power graphs of

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semigroups. Recently, many interesting results on the power graph of a finite group have been obtained (see [4], [5], [17], [18], [23]). A power graph is always connected. For other results and open questions, we refer the survey [1].

Let Γ be a finite, simple, and connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *distance* $d_\Gamma(u, v)$ between two vertices $u, v \in V(\Gamma)$ is the length of the shortest path between them. Let $W = \{w_1, w_2, \dots, w_k\}$ be an ordered set of vertices of Γ , and let $v \in V(\Gamma)$. Then, the *representation* of v with respect to W is the k -tuple $(d_\Gamma(v, w_1), d_\Gamma(v, w_2), \dots, d_\Gamma(v, w_k))$. Two vertices $u, v \in V(\Gamma)$ are resolved by W if they have different representations. W is *resolving set* or *locating set* if every vertex of Γ is, uniquely, identified by its distances from the vertices of W . Thus, in a resolving set, every vertex of Γ has distinct representations. A resolving set of minimum cardinality is called a *basis* for Γ . The cardinality of such a resolving set is called the *metric dimension* of Γ , and is denoted by $\beta(\Gamma)$ (see [3], [7], [19], [11], [20], [22]). A resolving set is called *minimal* if it contains no resolving set as a proper subset. As an application, S. Khuller [15] considered the metric dimension and basis of a connected graph in robot navigation problems.

Whenever W_1 and W_2 are any two minimal resolving sets for Γ , and for every $u \in W_1$ there is a vertex $v \in W_2$ such that $(W_1 \setminus \{u\}) \cup \{v\}$ is also a minimal resolving set. Then, Resolving sets are said to have *the exchange property* in the graph Γ (for detail, see [2]). All the graphs considered in this paper are finite, simple, and connected. Also, all the groups considered are finite. Furthermore, the exchange property of a graph Γ always means the property for resolving sets.

The *open neighborhood* of a vertex $u \in V(\Gamma)$, denoted by $N(u)$, is the set

$$\{v \in V(\Gamma) : d_\Gamma(u, v) = 1\},$$

and the *closed neighborhood* of $u \in V(\Gamma)$, denoted by $N[u]$, is the set

$$\{v \in V(\Gamma) : d_\Gamma(u, v) = 1\} \cup \{u\}.$$

The two vertices u and v in a graph Γ are called *twins*, denoted by $u \equiv v$, if either $N[u] = N[v]$ or $N(u) = N(v)$. The relation \equiv is an equivalence relation (see [10]). Also, $d_\Gamma(u_1, w) = d_\Gamma(u_2, w)$ for $u_1 \equiv u_2$, and for all $w \in V(\Gamma)$. Let \bar{u} denote the twin class of u with respect to the relation " \equiv ", and let $\mathcal{U}(\Gamma) = \{\bar{u} | u \in \Gamma\}$ be the set of all such twin classes.

Definition 1.1. A vertex u is called *singleton twin* if $\bar{u} = \{u\}$.

Our first result give a formula to compute the metric dimension of a graph, having no singleton twin.

Theorem 1.2. *Let Γ be a graph, having no singleton twin. Let there are n non-singleton twins, each of size m_i . Then,*

$$\beta(\Gamma) = \sum_{i=1}^n m_i - n.$$

Moreover, every minimal resolving set is a basis of the graph Γ .

Our second result provides a sufficient condition for a graph to have the exchange property.

Theorem 1.3. *A graph Γ , having no singleton twin, has the exchange property.*

In our next theorem, the metric dimension of the power graph, associated to a dihedral group D_{2n} of order $2n$, is computed.

Theorem 1.4. $\beta(\mathcal{P}_{D_{2n}}) = \beta(\mathcal{P}_{Z_n}) + n - 2$, where Z_n is a cyclic group of order n .

In the following theorem, we identified some finite groups whose corresponding power graph has the exchange property.

Theorem 1.5. *Let G be a finite group and \mathcal{P}_G be the power graph associated to G . Then, the exchange property holds in \mathcal{P}_G if one of the following conditions is satisfied:*

- (i) G is cyclic and $|G| = 2k + 1$, for positive integers k .
- (ii) $|G|$ is a power of a prime number p , and G is abelian.

The rest of the sections are organized as follows:

In section 2, the exchange property is discussed; also, proofs of Theorem 1.2 and 1.3 are given. In section 3, Theorems 1.4 and 1.5 are proved.

2 The Exchange Property for Resolving Sets

Every vector in a finite dimensional vector space is, uniquely, determined (written as a linear combination) by the elements of a basis of the vector space. A basis of a vector space has the exchange property. Similarly, each vertex of a finite graph can be, uniquely, identified by the vertices of a resolving set. Therefore, resolving sets of a finite graph behave like bases in a finite dimensional vector space. Unlike bases of a vector space, the

resolving sets do not always have the exchange property. The following results about the exchange property for different graphs can be found in the literature.

Theorem 2.1 ([2], Theorem 3). *The exchange property holds for resolving sets in trees.*

Theorem 2.2 ([2], Theorem 7). *For $n \geq 8$, the resolving sets do not have the exchange property in a wheel graph W_n .*

Theorem 2.3 ([21], Theorem 5). *For $n \geq 4$ and n is even, the necklace graph Ne_n does not have the exchange property.*

Lemma 2.4. *Let W be a resolving set for a graph Γ , and $v_1, v_2 \in V(\Gamma)$. Then, either $v_1 \in W$ or $v_2 \in W$ for $v_1 \equiv v_2$.*

Proof. For $v_1 \equiv v_2$, we have $d_\Gamma(v_1, u) = d_\Gamma(v_2, u)$ for all $u \in V(\Gamma) \setminus \{v_1, v_2\}$. Therefore, v_1 and v_2 can not be part of $V(\Gamma) \setminus W$. Otherwise, v_1 and v_2 remain unresolved. \square

Proof of the Theorem 1.2: Let W be a basis of Γ . By Lemma 2.4, W contains $m_i - 1$ vertices of each twin class of size m_i . Now, let u and v be two vertices which are not twins. Then, there must be some $w \in W$ such that $d_\Gamma(u, w) \neq d_\Gamma(v, w)$; otherwise $d_\Gamma(u, x) = d_\Gamma(v, x)$ for all $x \in V(\Gamma)$, which means that u and v are twins, a contradiction. Consequently, exactly one representative, from each twin class, stay outside W . Therefore, $\beta(\Gamma) = \sum_{i=1}^n m_i - n$.

The cardinality of a minimal resolving set W_1 is $\geq \beta(\Gamma)$. Now, W_1 must have exactly $\beta(\Gamma) = \sum_{i=1}^n m_i - n$ vertices. Otherwise, W_1 contains an entire twin class \bar{u} of a vertex u , and $W_1 \setminus \{u\}$ is again resolving set, a contradiction. Therefore, every minimal resolving set is a basis.

Proof of the Theorem 1.3: Let W_1 and W_2 be two different minimal resolving sets in the graph Γ , and let $u_1 \in W_1$. If $u_1 \in W_2$, then obviously $(W_1 \setminus \{u_1\}) \cup \{u_1\}$ is a minimal resolving set. Let $u_1 \notin W_2$. There exists a vertex $u_2 \notin W_1$ such that $u_1 \equiv u_2$. Otherwise, W_1 contains an entire twin class, and W_1 is not minimal by Theorem 1.2, a contradiction. Now, by Lemma 2.4, $u_2 \in W_2$, and every vertex in $V(\Gamma) \setminus \{u_1, u_2\}$ is at the same distance from the vertices u_1 and u_2 . Therefore, the vertices which are resolved by u_1 are also resolved by u_2 and vice versa. Consequently, $(W_1 \setminus \{u_1\}) \cup \{u_2\}$ is again a minimal resolving set. \square

3 Power Graph of Finite groups

Proposition 3.1 ([5], Proposition 2). *Suppose x and y are elements of an abelian group G , then x and y have the same closed neighborhoods, in the power graph \mathcal{P}_G , if and only if one of the following holds:*

- (i) $\langle x \rangle = \langle y \rangle$;
- (ii) G is cyclic, and one of x and y is a generator of G and the other is the identity e ; and
- (iii) G is cyclic of prime order (x and y are arbitrary).

Definition 3.2. ([8]) *For elements x and y in a group G , write $R\{x, y\} = \{z : z \in V(\mathcal{P}_G), d_{\mathcal{P}_G}(x, z) \neq d_{\mathcal{P}_G}(y, z)\}$.*

An involution is a non-identity element of order 2 of a group G . A resolving involution, in the power graph \mathcal{P}_G of a group G , is an involution w satisfies that, there exist two vertices $x, y \in V(\mathcal{P}_G) \setminus \overline{w}$ with $R\{x, y\} = \{x, y, w\}$. let $W(\mathcal{P}_G)$ denotes the set of all resolving involutions of \mathcal{P}_G .

Example 3.3. *Let $G = \{e, x, x^2, x^3, x^4, x^5\}$ be the cyclic group of order 6. Note that $R\{x, y\} = \{u, v, x^3\}$, for $u \in \{x, x^5\}$, and $v \in \{x^2, x^4\}$. Therefore, x^3 is a resolving involution of \mathcal{P}_G .*

Let Ψ denote the set of noncyclic groups G such that there exists an odd prime p such that the following conditions hold (see [8]):

- (C1) the prime divisors of $|G|$ are 2 and p ;
- (C2) the subgroup of order p is unique;
- (C3) there is no element of order 4 in G ; and
- (C4) each involution of G is contained in a cyclic subgroup of order $2p$.

In the original paper [8], for a finite group G , the notations $|G|$; $|\mathcal{U}(G)|$; and $|W(G)|$ were used for $|V(\mathcal{P}_G)|$; $|\mathcal{U}(\mathcal{P}_G)|$; and $|W(\mathcal{P}_G)|$ respectively. We give the following results in our notations.

Theorem 3.4. ([8], Theorem 3.23)

- (i) *If $G \in \Psi$, then*

$$\beta(\mathcal{P}_G) = |V(\mathcal{P}_G)| - |\mathcal{U}(\mathcal{P}_G)| + 1.$$

- (ii) *If $G \notin \Psi$, then*

$$\beta(\mathcal{P}_G) = |V(\mathcal{P}_G)| - |\mathcal{U}(\mathcal{P}_G)| + |W(\mathcal{P}_G)|.$$

Corollary 3.5 ([8], Corollary 3.24). *Suppose that $n = p_1^{r_1} \cdots p_t^{r_t}$, where p_1, \dots, p_t are primes with $p_1 < \cdots < p_t$, and r_1, \dots, r_t are positive integers. Let Z_n denote the cyclic group of order n . Then*

$$\beta(\mathcal{P}_{Z_n}) = \begin{cases} n-1, & \text{if } t=1; \\ n-2r_2, & \text{if } (t, p_1, t_1) = (2, 2, 1); \\ n-2r_1, & \text{if } (t, p_1, t_2) = (2, 2, 1); \\ n+1 - \prod_{i=1}^t (r_i + 1), & \text{otherwise.} \end{cases}$$

A dihedral group is presented as:

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, (ab)^2 = e \rangle.$$

D_{2n} is the disjoint union of the cyclic subgroup $Z_n \cong \langle a \rangle = \{e, a, a^2, \dots, a^{n-1}\}$, and the set of involutions $B = \{b, ab, a^2b, \dots, a^{n-1}b\}$.

Lemma 3.6. *Let $w \in B$, then, in the power graph $\mathcal{P}_{D_{2n}}$, the following are true:*

- (i) $\overline{w} = B$;
- (ii) w is not a resolving involution.

Proof. The neighborhood, in the graph $\mathcal{P}_{D_{2n}}$, of every involution $w \in B$ is $\{e\}$. Therefore, $\overline{w} = B$. If $x, y \in V(\mathcal{P}_{D_{2n}}) \setminus B$, then there are two possibilities:

- 1) $x = a^s, y = e, 1 \leq s \leq n-1$;
- 2) $x = a^{s_1}, y = a^{s_2}, 1 \leq s_1, s_2 \leq n-1$.

In the above two cases, one can see that $R\{x, y\} \neq \{x, y, w\}$. Therefore, $w \in B$ is not a resolving involution. \square

Proof of the Theorem 1.4: By part (ii) of Lemma 3.6, every resolving involution in $\mathcal{P}_{D_{2n}}$ belongs to the subgraph $\mathcal{P}_{\langle a \rangle}$, corresponding to the cyclic subgroup $\langle a \rangle$, of D_{2n} . Therefore, $W(\mathcal{P}_{D_{2n}}) = W(\mathcal{P}_{\langle a \rangle})$. In the subgraph $\mathcal{P}_{\langle a \rangle}$, the identity e and the generator a are twins. However, e is the unique singleton twin in $\mathcal{P}_{D_{2n}}$. By part (i) of Lemma 3.6, all $w \in B$ are in the same twin class. Therefore, the set $\mathcal{U}(\mathcal{P}_{D_{2n}})$ is the disjoint union of $\mathcal{U}(\mathcal{P}_{\langle a \rangle})$; the twin class of e ; and the twin class of w , for $w \in B$. Consequently, $|\mathcal{U}(\mathcal{P}_{D_{2n}})| = |\mathcal{U}(\mathcal{P}_{\langle a \rangle})| + 2$. A dihedral group D_{2n} does not satisfy the condition (C4); therefore, $D_{2n} \notin \Psi$. Now, put $|V(\mathcal{P}_{D_{2n}})| = |V(\mathcal{P}_{\langle a \rangle})| + n$; $|\mathcal{U}(\mathcal{P}_{D_{2n}})| = |\mathcal{U}(\mathcal{P}_{\langle a \rangle})| + 2$; and $|W(\mathcal{P}_{D_{2n}})| = |W(\mathcal{P}_{\langle a \rangle})|$ in the equation of part (ii) of Theorem 3.4 to complete the proof. \square

To compute the exact value of $\beta(\mathcal{P}_{D_{2n}})$, one can use Theorem 1.4 and Corollary 3.5.

Lemma 3.7. *A singleton twin x , in the power graph \mathcal{P}_G , is either an involution or the identity e in the group G .*

Proof. If x in G is not an involution or e , then the order $o(x)$ is ≥ 3 and $N[x] = N[x^{-1}]$, a contradiction. \square

Proof of the Theorem 1.5: Let G be a cyclic group of odd order, and y is a generator of G . Then, there is no involution in the group G . Also, part (ii) of Proposition 3.1 implies that $\overline{y} = \overline{e}$, and e is not a singleton twin. Therefore, by Lemma 3.7, the graph \mathcal{P}_G has no singleton twin. Hence, by Theorem 1.3, the exchange property holds in \mathcal{P}_G .
Let G be an abelian group of order $|G| = p^m$, for some prime p , then \mathcal{P}_G is a complete graph. Therefore, \mathcal{P}_G has no singleton twin, and has the exchange property by Theorem 1.3.

The following example shows that the converse of Theorem 1.3 and Theorem 1.5 is not true.

Example 3.8. *Let \mathcal{P}_{Z_6} be the power graph, where $Z_6 \cong \langle x \rangle = \{e, x, x^2, x^3, x^4, x^5\}$. Then, the order of the group is even and not a power of a prime. Furthermore, the power graph contains the singleton twin x^3 . Still, the graph has the exchange property for resolving sets.*

We did not encounter a power graph of a finite group which does not have the exchange property. Therefore, the following question make sense to be posed.

Question 3.9. *Does there exist a finite group whose power graph does not hold the exchange property?*

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